Complex Cobordism and Formal Group Law

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Abstract

This thesis contains two parts. In the first part (sections 1 - 3), we prove a conjecture by Quillen, for which he gave a sketch of proof in his 1969 notes ([6]), that relates the calculation of Gysin homomorphism in complex cobordism to the residue symbol; the interesting part is in section 3. In sections 1 and 2, I collect basic results about complex cobordism and the formal group law in addition to giving a proof that complex cobordism satisfies Poincaré duality. In the second part (section 4), calculations and a recursive formula are given to the correction term to the virtual fundamental class in the definition of Gromov-Witten invariants in complex-oriented generalised cohomology theories ([1]), which gives rise to an associative quantum product.

Contents

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I am indebted to the late Dr. Jonathan Richmond, without whose generous guidance and mentorship, I would certainly have not been able to study at Stanford, and my life would have turned out very differently.

1 Complex cobordism

In this section, we recall some definitions and properties about stable complex structures and prove at the end of the section that the generalized cohomology theory of complex cobordism satisfies Poincaré duality.

1.1 Stable complex structures

We first recall the definition of stable complex structures.

Definition 1.1. *A manifold M is stably complex if it is equipped with a complex structure on the bundle $TM \oplus \mathbb{R}^k$, the direct sum of its tangent bundle and a k-dimensional trivial real line bundles. A stable complex structure on M is determined by the choice of the complex structure of $TM \oplus \mathbb{R}^k$.*

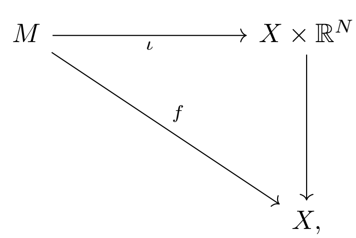
We will introduce a notion of stable normal complex structures later, and to distinguish the stable complex structure from the stable normal complex structure, we will sometimes refer to the stable complex structure defined in (1.1) as the tangential complex structure. We can define a notion of equivalence for stable complex structures. Two stable complex structures over M, i.e. complex structures on $TM \oplus \mathbb{R}^k$ and $TM \oplus \mathbb{R}^l$ for some non-negative integers k and l, are equivalent if there exists non-negative integers a and b such that the complex structures on $TM \oplus \mathbb{R}^k \oplus \mathbb{C}^a$ and $TM \oplus \mathbb{R}^l \oplus \mathbb{C}^b$ induced from the original stable complex structureds and complex multiplication on the $\mathbb{C}$ summands are the restriction of a complex vector bundle $E \to M \times [0,1]$ to endpoints.

More generally, we say that two complex vector E and E' bundles over the same base space X are stably equivalent if there exists a vector bundle $E'' \to X \times [0,1]$ such that the direct sum of E and E' with some trivial complex vector bundles agree with the restriction of E'' to $X \times 0$ and $X \times 1$, respectively.

1.2 Complex orientation

We want to define the notion of a complex-oriented map, which generalizes the notion of a stable complex structure to that of a stable normal complex structure for a map $f : M \to X$, so in particular the stable complex structure in the previous section can be recovered from the stable normal complex structure of $M \to pt$ with the target manifold set to a point. We then define a complex orientation as an equivalence relation of complex-oriented maps, which will be important later when we define complex cobordism geometrically as a generalized cohomology theory.

Definition 1.2. *A complex-oriented map $f : M \to X$ is a choice of the complex structure on the normal bundle associated with a factorization of f through the trivial bundle $\pi : X \times \mathbb{R}^N \to X$*



where $\iota : M \to X \times \mathbb{R}^N$ is an embedding with the choice of the complex structure on the normal bundle $N_\iota M$ associated with ι.

Recall that by a proper map, we mean that the preimage of a compact set is compact. We see that a proper complex-oriented map is preserved under pullback.

Theorem 1.1. *Suppose that we are given a proper complex-oriented map $f : Z \to X$ and a map $g : Y \to X$ between smooth manifolds that are transversal, then the pullback $f^* : Y \times_X Z \to Y$ is again a proper complex-oriented map.*

Proof. First, notice that the pullback of a proper map is proper. Fix an embedding $\iota : Z \to X \times \mathbb{R}^n$ that f factors through and let $g' : Y \times \mathbb{R}^n \to X \times \mathbb{R}^n$ be the map induced by g on the first factor and the identity map on $\mathbb{R}^n$, then g' intersects transversely with ι because g and f intersect transversely. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Z \\ \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} \\ Y \times \mathbb{R}^n & \xrightarrow[g']{} & X \times \mathbb{R}^n, \end{array}$$

which is Cartesian (where ι' is an embedding that lifts the pullback $Y \times_X Z \to Y$) and gives an isomorphism between the normal bundle associated to ι and ι'. This defines a complex oriented map $f^* : Y \times_X Z \to Y$. ☐

There is a notion of cobordism between complex oriented maps that generalizes the cobordism relation of manifolds.

Definition 1.3. *Two proper complex-oriented maps $f_i : Z_i \to X$, $i = 0, 1$, are called **cobordant** if there exists a proper complex-oriented map $b : W \to X \times \mathbb{R}$ such that the maps $\epsilon_i : X \to X \times \mathbb{R}$, where $i = 0, 1$ and $\epsilon_i(x) = (x, i)$, are transversal to the map b and the, pullback of b by ϵ_i is isomorphic to the complex-oriented map f_i, i.e. the following the commutative diagram has cartesian squares.*

$$\begin{array}{ccccc} Z_0 & \longrightarrow & W & \longleftarrow & Z_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle f_1} \\ X & \xrightarrow{\epsilon_0} & X \times R & \xleftarrow{\epsilon_1} & X \end{array}$$

In particular, if one of the Z_i is empty, e.g. if Z_0 is a non-empty manifold and Z_1 is empty, then we say that Z_0 is null-cobordant if there exists such a $W \to X \times \mathbb{R}$ such that the left square in the preceding commutative diagram is Cartesian.

We can define addition on the set cobordant complex oriented maps by disjoint union. Specifically, given complex-oriented maps $f : M \to X$ and $g : Z \to X$, their sum is $f + g : M \bigsqcup Z \to X$. The zero element is represented by the empty set, and the inverse of $-f$ of a map $f : M \to X$ is given by f with the opposite complex orientation (let J be a complex structure, its opposite complex orientation is the complex structure $-J$) in the normal bundle of the embedding, because we can form a complex oriented map $b : W = Y \times \mathbb{R} \to X \times \mathbb{R}$ such that the restriction of its associated normal bundle at $Y \times 0$ and $Y \times 1$ agrees with the complex oriented maps f and f'.

We want to make our definition of complex-oriented maps compatible with the homotopy class of complex structures. This motivates to define the following equivalence relation.

Definition 1.4. *Two complex oriented maps from M to X are equivalent if their associated normal bundles are stably equivalent. By a complex orientation, we mean an equivalence class of complex oriented maps.*

More specifically, two complex oriented maps $f : M \to X$ and $f' : M \to X$ with embeddings $\iota : M \to X \times \mathbb{R}^n$ and $\iota' : M \to X \times \mathbb{R}^m$ are equivalent if we can form new embeddings $\tilde{\iota} : M \to X \times \mathbb{R}^n \times \mathbb{C}^k$ and $\tilde{\iota'} : M \to X \times \mathbb{R}^m \times \mathbb{C}^l$ (where l and k are non-negative integers) such that they induced cobordanct complex-oriented maps.

By a stable complex normal structure for the map $f : M \to X$, we mean a choice of a representative of a complex orientation. The following theorem shows that if the target manifold X is stably complex for a map $f : M \to X$, then the choice of a stable complex structure up to equivalence on M is equivalent to the choice of a complex orientation.

We now recall a version of the Whitney approximation theorem [4].

Theorem 1.2. *(Whitney approximation theorem) Let M be a smooth manifold and let $F : M \to \mathbb{R}^k$ be a continuous map. Given a positive continuous function $\delta : M \to \mathbb{R}$, there exists a smooth map $\tilde{F} : M \to \mathbb{R}^k$ that is δ-close to F (in the sense that the Euclidean distance between the two maps at each point is bounded by delta). If F is smooth on a closed subset $A \subset M$, then $\tilde{F}$ can be chosen to be equal to F on A.*

Note that by the Whitney approximation theorem, we can always find such an embedding $\iota : M \to X \times R^N$ for N sufficiently large by approximating the continous map $M \to X \times R^N$ which maps M to the zero section and lifts f. The choice of such a stable complex normal structure depends on the specific embedding, so we define a notion of equivalence. Let N_ι and $N_{\iota'}$ be the normal bundles defined this way associated to two emeddings $\iota : M \to X \times \mathbb{R}^N$ and $\iota' : M \to X \times \mathbb{R}^{N'}$ and assume they define stable complex normal strucutre J and J'.

Theorem 1.3. *For a map $f : M \to X$ into a stably complex manifold X, there is a 1-1 correspondence between the equivalence classes of stable complex structures on M and the complex orientations associated with f.*

Proof. The idea of the proof is that we given a stable complex structure for M, we want to use the stable complex structure of X to induce a stable normal complex structure for f and vice versa.

Suppose $TM \oplus \mathbb{R}^k$ carries an almost complex structure, then we fix an embedding $\iota : M \to X \times \mathbb{R}^N$ that lifts f (such an embedding always exists by the Whitney embedding theorem) and let $N_\iota M$ be the associated normal bundle. Notice that $N_\iota M \oplus TM \oplus \mathbb{R}^k = TX|_M \oplus \mathbb{R}^{N+k}$. We can define a complex structure J on $N_\iota M \oplus N_\iota M$ by $J(a, b) = (-b, a)$. This gives an almost complex structure on the bundle

$$\begin{aligned} E_\iota(M) &= (N_\iota M \oplus N_\iota M) \oplus (TM \oplus \mathbb{R}^k) \\ &= N_\iota M \oplus (N_\iota M \oplus TM \oplus \mathbb{R}^k) \\ &= N_\iota M \oplus TX|_M \oplus \mathbb{R}^{N+k}. \end{aligned} \tag{1}$$

Although the bundle $E_\iota(M)$ depends on the choice of the embedding ι it is indepdent of the choice of the embedding in the following sense. Suppose $\iota' : M \to X \times \mathbb{R}^{N'}$ is another embedding lifting f, then let N'' be a sufficently large number to be chosen later, so that $X \times \mathbb{R}^N$ and $X \times \mathbb{R}^{N'}$ embed in $X \times \mathbb{R}^{N''}$ as sub-vector bundles. Consider a continuous map $g : M \times [0, 1] \to X \times \mathbb{R}^{N''}$, such that $g|_i$ extends ι_i for $i = 0, 1$, $g_{\frac{1}{2}}$ maps into the zero section and agrees with f, and for $t \in (0, 1)$, g_t is defined by a linear homotopy between the image of $g_{\frac{1}{2}}$ and g_i for $i = 0, 1$. For N'' sufficiently large satisfying the condition of Whitney approximation (1.2), we can make g homotopic to a smooth embedding that agree on neighbourhoods of $M \times \{i\}$. The complex structures on $E_\iota M$ and $E_{\iota'} M$ are induced by restriction of $E_{\iota''}(M \times [0, 1])$ up to direct sum with some trivial complex vector bundles.

If X is a point, then $E_\iota(M)$ is just $N_\iota M \oplus \mathbb{R}^{N+k}$, which can be identified with a complex normal bundle for some embedding of M into $X \times \mathbb{R}^{2N+k}$, and from discussion in the previous paragraph, different choices of the embedding give cobordant complex structures on the normal bundle (after potentially passing to direct sum with trivial $\mathbb{C}$ summands), so the complex orientation induced is independent of the choice of the embedding.

Let $T(M)$ and $N_f(M)$ be the set of tangential and normal stable complex structures for a map $f : M \to X$ (recall the stable tangential complex structure depends just on the source manifold M), respectively, and let $\tilde{T}(M)$ and $\tilde{N}_f(M)$ be the sets of their equivalence classes. For each element in $T(M)$, we can form

$E_\iota M \oplus N_{pt}X$ which represents an element in $\tilde{N}_f(M)$ and whose equivalence class is independent of the chosen embedding. Notice the assignment is independent of the choice of equivalent stable tangential complex structure, since equivalence of the tangential complex structures extend to the equivalence of the bundles defined by $E_\iota(M)$. So this give a map $\phi : \tilde{T}(M) \to \tilde{N}_f(M)$.

If we exchange the roles of $TM \oplus \mathbb{R}^k$ and $N_\iota M$, we also get a map $\psi : \tilde{N}_f(M) \to \tilde{T}(M)$. Specifically, given an almost complex structure for the normal bundle $N_\iota M$ for an embedding ι, we can define an almost complex structure on a new bundle

$$\begin{aligned} \psi(N_\iota M) &= (TM \oplus TM) \oplus (N_\iota M) \oplus N_{pt}X \\ &= TM \oplus \mathbb{R}^l \end{aligned} \tag{2}$$

for some non-negative integer l which defines a stable normal complex struucture, and one can check as before that ψ defines a map from $\tilde{N}_f(M)$ to $\tilde{T}(M)$.

We check that the associations ψ and ϕ are inverse of each other. We do this for showing $\psi \circ \phi$ is the identity map on $\tilde{T}(M)$ as the other case is similar. Indeed, as an almost complex structure on $TM \oplus \mathbb{R}^k$, $\psi \circ \phi$ associates an almost complex structure

$$\begin{aligned} E &= (TM \oplus TM) \oplus (N_\iota M \oplus N_\iota M \oplus TM \oplus \mathbb{R}^k) \oplus N_{pt}X \oplus N_{pt}X \\ &= (TM \oplus \mathbb{R}^k) \oplus (TM \oplus N_\iota M \oplus N_{pt}X) \oplus (TM \oplus N_\iota M \oplus N_{pt}X) \\ &= (TM \oplus \mathbb{R}^k) \oplus \mathbb{C}^a \oplus \mathbb{C}^a \end{aligned} \tag{3}$$

which is the direct sum of $TM \oplus \mathbb{R}^k$ with two trivial complex vector bundles, so E is equivalent to $TM \oplus \mathbb{R}^k$. □

1.3 Complex cobordism

The k^{th} homotopic complex cobordism ring can be defined as a direct limit

$$MU^k(X) = \lim_{n \to \infty} [\Sigma^{2n-k} X \to MU(n)], \tag{4}$$

where MU is the Thom spectrum and and ΣX is the suspension of X. We want to give it a geometric interpretation and identify the homotopic complex cobordism ring $MU^k X$ with the set of cobordant complex orientations.

A complex orientation $f : M \to X$ has codimension $dim(X) - dim(M)$. Let $C_n(X)$ denote the set of complex oriented maps of codimension n to a target manifold X, and let $\Sigma^n(X)$ denote the set of equivalent classes in $C_n(X)$ up to cobordism. We have the following theorem.

Theorem 1.4. *The complex cobordism ring $MU^n(X)$ is isomorphic to $\Omega^n(X)$.*

Proof. We first define a map $\phi : C_n(X) \to MU^n(X)$. Suppose we are given a map $f : M \to X \in C_n(X)$ with an embedding $\iota : M \to X \times \mathbb{R}^N$ and let $N_\iota(M)$ be the associated normal bundle identified as a tubular neighbourhood in $X \times \mathbb{R}^N$; $N_\iota(M)$ carries an almost complex structure. As in [9], we have the following maps in the commutative diagram

$$\begin{array}{ccc} N_\iota M & \xrightarrow{\ i\ } & EU(k) \\ \downarrow & & \downarrow \\ M & \longrightarrow & BU(k) \end{array}$$

where $EU(k) \to BU(k)$ is the ball bundle induced by the universal bundle classifying almost complex structures of dimension k, and $k = \frac{N+dim(X)-dim(Y)}{2} = \frac{N+n}{2}$. If we view $N_\iota M$ as a ball bundle via the tubular neighourhood theorem, the map fi carries its boundary to the boundary of $EU(k)$, inducing a map $i : N_\iota M/\partial N_\iota M \to EU(k)/\partial EU(k) = MU(k)$. This induces a map $g : X \times \mathbb{R}^N \to MU(k)$, where we map the complement of $N_\iota M$ in X to the point in $MU(k)$ identified with $\partial EU(k)$. By the properness of f, this induces a map $g : X \times \mathbb{S}^N/X \times pt = \Sigma^N X \to MU(k)$ by viewing $\mathbb{S}^n$ as the one-point compactification of $\mathbb{R}^N$.

The map g, viewed as an element of $\lim_{n\to\infty}[\Sigma^{2n-k}X \to MU(n)]$, is independent of the choice of the stable normal complex structure. To see this, suppose that g' and g represent embeddings for two equivalent stable normal complex structures and notice taking direct sum with a trivial complex line bundle corresponds to taking the suspension of the complex oriented map and composing it with the canonical map $\Sigma MU(n) \to MU(n+1)$. If a complex oriented map $F : M \times [0,1] \to X$ with an embedding $\iota_F : M \times [0,1] \to X \times \mathbb{R}^{n'}$ induces the complex normal bundles associated with g and g' (possibly after taking direct sum with some trivial complex bundles) on ends, then for each $t \in [0,1]$, F_t induces a map (as previously defined) $f_t : \Sigma^{2n'-k}X \to MU(n')$ for some fixed n' which is independent of t. This gives a homotopy between suspensions of g and g'. Hence we have a well-defined map $\phi : C_n(X) \to MU^n(X)$.

The map ϕ assigns cobordant complex oriented maps to the same homotopy class. The proof is very similar to how we showed that ϕ is well-defined for equivalent complex oriented maps, for if $M_i \to W$ and $b : W \to X \times \mathbb{R}$ corresponds to the cobordance for $i = 0$ or 1, then $\phi(b)$ gives the desired homotopy between $M_i \to X$.

We can apply Thom's argument in [9] directly to show ϕ is surjective. Any element $g \in [\Sigma^{2n-k}X \to MU(n)]$ can be homotopied to be transversal to $BU(n) \subset MU(n)$ and hence defines a complex vector bundle over the submanifold $M = g^{-1}(BU(n))$ in $\Sigma^{2n-k}X$. This gives an embedding of M in $X \times \mathbb{R}^n$ and an almost complex structure on the normal bundle, giving rise to a complx oriented map $f : M \to X$ such that $\phi(f) = g$

The map ϕ is also injective, because suppose $f_i : M_i \to X$ maps to the same element $g \in [\Sigma^{2n-k}X \to MU(k)]$ in the stable range, so there exists a homotopy $F : \Sigma^{2n-k}X \times [0,1] \to MU(k)$, and by Thom's transversality theorem we can move F by a homotopy fixing endpoints so that F is transversal to $BU(k) \subset MU(k)$. Using the construction in the previous paragraph, we can construct a submanifold $W = F^{-1}(BU(k))$ giving the required cobordism relation. Therefore, the map ϕ is a bijection.

□

1.4 Ring structure

We can define a commutative ring structure on the complex cobordism ring $\Omega^*(X)$, which is identified with the set of complex oriented maps up to complex cobordism. First, we show that complex cobordism is a contravariant functor.

Theorem 1.5. $\Omega^*(-)$ *is a contravariant functor from the category of smooth manifolds to sets.*

Later we will lift the target category to the category of commutative rings.

Proof. Suppose $f : M \to X$ is a complex oriented map with the following factorization

$$\begin{array}{ccc} M & \xrightarrow{\iota} & X \times \mathbb{R}^N \\ & \searrow^{f} & \downarrow \\ & & X \end{array}$$

and $g : Y \to X$ is a map, then by Thom's transversality theorem, we can deform f and assume f intersects transversaly with g. Consider the fiber product

$$\begin{array}{ccc} Y \times_X M & \longrightarrow & M \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\quad g \quad} & X. \end{array}$$

Recall that the pullback $f' : Y \times_X M \to Y$ has a complex orientation

$$\begin{array}{ccc} Y_X M & \xrightarrow[\iota']{\qquad} & Y \times \mathbb{R}^N \\ & \searrow & \downarrow \\ & & Y \end{array}$$

induced from the complex orientation of $f : M \to X$, where concretetly $\iota' : Y \times_X M \to Y \times \mathbb{R}^N$ is given by $\iota'(y, m) = (y, \pi_{\mathbb{R}^N} \cdot \iota(m))$.

We need now to show that the cobordism class of f' does not depend on the choice of a representative $f : M \to X$ or the way we perturb f by homotopy. To see that the pullback does not depend on the choice of the homotopy, observe that a homotopy that perturbs g induces a map $G : Y \times [0, 1] \to X$ which can be made transversal to f while fixing G on endpoints. We can form the fiber product of G and f which gives is homeomorphic to $Y \times_X M \times [0, 1]$. The restriction of the stable normal complex structure on $Y \times_X M \times 0$ and $Y \times_X M \times 1$ agrees with the stable normal complex structures induced by $G(-, 0)$ and $G(-, 1)$.

It suffices now to show that two representative f_0 and f_1 in the same cobordant class give rise to the same pullback up to cobordism.

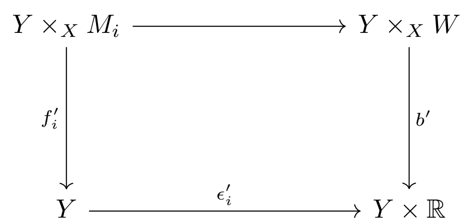
Suppose two maps $f_0 : M_0 \to X$ and $f_1 : M_1 \to X$ are cobordant and intersect transversely with g, respectively, so the following commutative diagram exists

$$\begin{array}{ccccc} M_0 & \xrightarrow{\quad i_0 \quad} & W & \xleftarrow{\quad i_1 \quad} & M_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle f_1} \\ X & \xrightarrow{\quad \epsilon_0 \quad} & X \times \mathbb{R} & \xleftarrow{\quad \epsilon_1 \quad} & X. \end{array}$$

Consider the following commutative diagram

$$\begin{array}{ccc} Y \times_X W & \longrightarrow & W \\ {\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle b} \\ Y \times \mathbb{R} & \xrightarrow[g \times id]{\qquad} & X \times \mathbb{R}l \end{array}$$

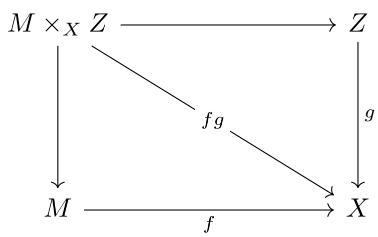
By Thom's transversality theorem and the fact that the preimage of $X \times i$ by b in W looks locally like the product manifold $M_i \times \mathbb{R}$ for $i = 1$ or 0, we can deform $g \times id$ so that it intersects transversely with b while fixing a neighbourhood of $g \times 0$ and $g \times 1$ which already intersect transversely with b. We can hence form the fiber product and show that the diagram above is cartesian using a Yoneda argument. Moreover, we also have that the following commutative diagram



is cartesian and intersects transversaly, which gives a cobordism relation to $Y \times_X M_i \to Y$.

□

One defines multiplication on $\Omega^*(X)$ in the following way: given $f : M \to X$ and $g : Z \to X$, we define fg as the pullback of $f \times g : M \times Z \to X \times X$ along the diagonal $\Delta : X \to X \times X$, so concretely $fg : M \times_X Z \to X$ is the fibre product of f and g as a smooth map, and the associated complex orientation corresponds to taking Whitney sum of the complex normal bundles for f and g.



Multiplication on $\Sigma^*(X)$ is commutative because the diagonal map in the preceding commutative diagram is the same if we swap f with g and taking Whitney sum of vector bundles is commutative. In particular the multiplicative identity is the identity map of X.

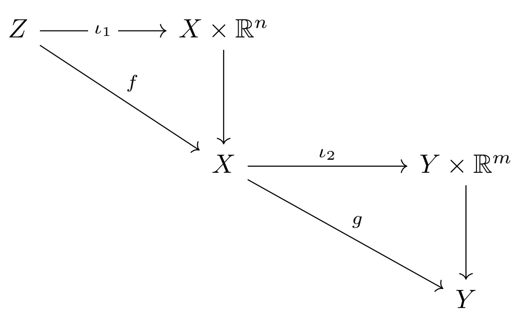
It is straightforward to see that the addition operation defined previous for complex-oriented maps defines an addition on complex orientations, so we have shown the complex cobordism ring $\Omega^*(X)$ is a commutative ring with unity.

One can also view the ring $\Omega^*(X)$ as an $\Omega^*(pt)$-module via the map $f^* : \Omega^*(pt) \to \Omega^*(X)$ induced by f that maps X to a point.

1.5 Gysin homomorphism

Given a complex oriented map $g : X \to Y$ of degree d, for any complex oriented map $f : Z \to X$ of degree n, we can form $g^! f : Z \to Y$ of degree $n - d$ by the following procedure.

Let the commutative diagram



represent the factorization associated to the complex-oriented maps g and f, so $N_{\iota_1} \oplus TZ = TX \oplus \mathbb{R}^n|_Z$ and $N_{\iota_2} \oplus TX = TY \oplus \mathbb{R}^m|_X$, we see that this induces a complex normal bundle $N_{\iota_1} \oplus f^*(N_{\iota_2})$ of an embedding $Z \to Y \oplus \mathbb{R}^{n+m}$, which defines a complex-oriented map $g^! f$.

It is straightforward to see that the cobordance class of $g^* f$ depends just on the corbodance classes of f and g, so this defines a Gysin homomorphism $g^! : \Omega^n(X) \to \Omega^{n-d}(Y)$.

Theorem 1.6. *The Gysin homomorphism commutes with pullback.*

Proof. Consider

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[h']{} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow[h]{} & Y, \end{array}$$

where the lower and upper sqaures are Cartesian, so the outer square is also Cartesian. The pullback of $g^! f$ the image of the Gysin homomorphism agrees as a topological map with the $g'^! f'$ the image of the Gysin homomorphism along g' of f' the pullback of f along h', and it is straightforward to check from definitions that they carry the same complex orientation. □

1.6 Poincaré duality

Recall that the homology group $\Omega_q(X)$ [2] consists of maps from stably complex manifolds of dimension q to X up to cobordism. We can define a cap product $\cap : \Omega_p(X) \otimes \Omega^q(X) \to \Omega_{p-q}(X)$ in the following way. Let $a \in \Omega_p(X)$ and $b \in \Omega^q(X)$, then $a \cap b = a_* a^*(b)$ the image under the Gysin hormormophism of the pullback of b along a. In particular, we can define the fundamental class as $a = 1$ the identity map on X with trivial complex orientation, then $a \cap b$ is just the same map b with the stable complex tangential structure induced by the complex orientation. Thus by results from the previous sections, we have the following theorem of Poincaré duality.

Proposition 1.1. *(Poincaré duality) The map* $1 \cap (-) : \Omega^q(X) \to \Omega_{n-q}(X)$ *is an isomorphism, where* $dim(X) = n$.

2 Formal group law

In this section, we collect some useful results about the formal group law, which will be useful in the next section when we compute the Gysin homomorphism of the projection map from a projective bundle to the base space.

2.1 Background

We refer the reader to Ravenel's "green book" [8] for basic definitions of the formal group law. The following results are collected from [8].

Definition 2.1. *Let* F *and* G *be formal group laws over* R. *A homomorphism from* F *to* G *is a power series* $f(x) \in R[[x]]$ *with constant term* 0 *such that* $f(F(x, y)) = G(f(x), f(y))$. *It is an isomorphism if* f *is invertible.*

Notice that the condition of being an isomorphism is equivalent to the condition that $f'(0)$ (the coefficient of x) is a unit in R, since we can write down the coefficients of the inverse recursively from lower degree terms. We call an isomorphism a strict isomorphism if $f'(0) = 1$. A strict isomorphism from F to the addition formal group law $x + y$ is a logarithm for F, denoted by $Log_F(x)$.

In general, a logarithm does not necessarily exist for an arbitrary formal group law, but the following theorem shows that a logarithm exists for a formal group law over a ring that is a $\mathbb{Q}$-algebra.

Theorem 2.1. *Let F be a formal group law and let R be a commutative ring such that $R \otimes \mathbb{Q}$ is not 0. Let $f(x) \in R \otimes \mathbb{Q}[[x]]$ be given by*

$$f(x) = \int_0^x \frac{dt}{\frac{\partial F}{\partial y}(t,0)}, \tag{5}$$

where $\int_0^x \sum_{i\geq 0} a_i t^i = \sum_{i\geq 0} \frac{a_i}{i+1} x^{i+1}$ denotes the formal anti-derivative. Then f is a logarithm for F, i.e., $F(x,y) = f^{-1}(f(x)+f(y))$. Moreover, the quantity

$$\frac{dt}{\frac{\partial F}{\partial y}(t,0)} \tag{6}$$

is called the invariant differential form.

Proof. First, we show that $f(x)$ actually defines a formal power series with leading term x. Let $\frac{\partial F}{\partial y}(t,0) = \sum_{t\geq 0} a_i t^i$ be the formal power series expansion, notice $a_0 = 1$ because $F(x,0) = F(0,x) = 0$, we can find a formal power series expansion

$$\frac{1}{\frac{\partial F}{\partial y}} = \sum_{i\geq 0} b_i x^i, \tag{7}$$

where the coefficient $\{b_i\}$ are defined inductively by $b_0 = 1$ and $b_i = a_i - \sum_{j=0}^{i-1} b_j a_{i-j}$, so explicityly $f(x) = \sum_{i\geq 0} \frac{b_i}{i+1} x^{i+1} \in R \otimes \mathbb{Q}[[x]]$ is a strict isomorphism.

Now we show $f(x)$ defines a logarithm, i.e. a strict isomorphism to the addition formal group law. Let $\omega(x,y) = f(F(x,y)) - f(x) - f(y)$, it suffices to show $\omega(x,y) = 0$. Note since $\omega(0,0) = 0$, it suffices to show the partial derivatives $\frac{\partial \omega}{\partial x}$ and $\frac{\partial \omega}{\partial y}$ vanish because $R \otimes \mathbb{Q}$ has characteristic 0.

By symmetry of exchanging x and y, it suffices to show that $\frac{\partial \omega}{\partial y}$ is identically 0. Diferentiating ω with respect to y gives

$$\begin{aligned} \frac{\partial \omega}{\partial y} &= f'(F(x,y)) \frac{\partial F}{\partial y}(x,y) - f'(y) \\ &= \frac{\frac{\partial F}{\partial y}(x,y)}{\frac{\partial F}{\partial y}(F(x,y),0)} - \frac{1}{\frac{\partial F}{\partial y}(y,0)} \\ &= \frac{\frac{\partial F}{\partial y}(x,y)}{\frac{\partial F}{\partial y}(x,y)} - \frac{1}{\frac{\partial F}{\partial y}(y,0)} \\ &= 0, \end{aligned} \tag{8}$$

where for the second last step, we use the fact that $F(x,0) = F(0,x) = x$ for a formal group law. ☐

2.2 Universal formal group law

The category of formal group laws has an initial object, as explained by the following theorem.

Theorem 2.2. *(Lazard) There is a ring L (called the Lazard ring) and a formal group law*

$$F(x,y) = \sum a_{i,j} x_i y_j \tag{9}$$

defined over L such that for any formal group law G over any commutative ring R there is a unique ring homomorphism $\theta : L \to R$ such that $G(x,y) = \sum \theta(a_i, j) x_i y_j$.

The construction invovles forming a free algebra over $\mathbb{Z}$, where each generator corresponds to a coefficient of the power series representing the universal formal group law, and quotienting by the relations defining a formal group law. For more details, refer to [8].

Using properties of the logarithm, we can deduce the following result.

Theorem 2.3. *(Lazard)*

- $L \otimes Q = Q[m_1, m_2, ...]$, *where we associate with a grading* $|m_i| = 2i$ *and* $F(x,y) = f^{-1}(f(x)+f(y))$ *where* $f(x) = x + \sum_{i>0} m_i x^{i+1}$.
- *Let* $M \subset L \otimes Q$ *be* $Z[m_1, m_2, ...]$. *Then* $\operatorname{im} L \subset M$, *where* $\operatorname{im} L$ *is the image of the isomrphism* $L \otimes Q = Q[m_1, m_2, ...]$.

Proof. Define $\phi : Q[m_1, m_2, ...] \to L$ by the unique homomorphism such that $\phi(f(x)) = log(x)$, where log is the logarithm of $L \otimes Q$, which exists because $L \otimes Q$ is not 0 (look at the generators). Notice $f(x)$ is the logarithm of the formal group law $F(x,y) = f^{-1}(f(x)+f(y))$ defined over $Q[m_1, m_2, ...]$. We define $\theta : L \otimes Q \to Q[m_1, m_2, ...]$ by tensoring with the map coming from the universal property of L.

We first show $\theta\phi$ is the identity on $L \otimes Q$. Notice that the universal formal group law F_L on L (which can be extended to $L \otimes Q$ under the inclusion of L) can be written as $F_L(x,y) = log^{-1}(log(x)+log(y))$, and θ is defined such that $\theta F_L = F$, so we have have

$$\begin{aligned} \theta F_L(x,y) &= (\theta log)^{-1}((\theta log)(x) + (\theta log)(y)) \\ &= F(x,y) \\ &= f^{-1}(f(x)+f(y)) \end{aligned} . \tag{10}$$

Since $\phi(f(x)) = log(x)$, we have

$$\begin{aligned} \phi\theta F_L(x,y) &= \phi(f^{-1}(f(x)+f(y))) \\ &= (\phi f)^{-1}((\phi f)(x) + (\phi f)(y)) \\ &= log^{-1}(log(x)+log(y)) \\ &= F_L(x,y) \end{aligned} , \tag{11}$$

so $\phi\theta$ induces the identity map on the formal group law F_L. Consider the restriction $\phi\theta : L \to L \otimes Q$, which induces the same formal group law F_L on $L \otimes R$ under the inclusion map, so by the universal property of L, $\phi\theta$ is the identity map, and $\theta(log(x)) = f(x)$. Hence, $\theta\phi(f(x)) = \phi(log(x)) = f(x) = x + \sum_{i>0} m^i x^{i+1}$. Since the set m_i generates $Q[m_1, m_2, ...]$, we have that $\theta\phi$ is the identity of $Q[m_1, m_2, ...]$, so we are done for the first part of the proof.

The second part follows by observing that $F(x,y)$ is a power series with coefficients in $Z[m_1, m_2, ...]$ and that ϕ is the unique homormorphism mapping the formal group law F_L to F.

□

Moreover, the ring L is actually a free polynomial ring, and a precise structure theorem is expressed by [8].

Theorem 2.4.

- $L = Z[x_1, x_2, ...]$ *with the grading* $|x_i| = 2i$ *for* $i > 0$.
- x_i *can be chosen so that its image in* $QL \otimes Q$ *is*

$$\begin{cases} pm_i, & \textit{if } i = p^k - 1 \textit{ for some prime } p \\ m_i, & \textit{otherwise.} \end{cases} \tag{12}$$

- *L is a subring of M* (2.3).

The ring L is the initial object in the category of formal group laws in a sense that will be defined next. We first define a notion of the matched pair.

Definition 2.2. *Let R be a commutative ring with unit. Then $FGL(R)$ is the set of formal group laws over R and $SI(R)$ is the set of triples (F, f, G) where $F, G \in FGL(R)$ and $f : F \to G$ is a strict isomorphism. We call such a triple a* ***matched pair***.

Theorem 2.5. *(Representability) $FGL(-)$ and $SI(-)$ are covariant functors on the category of commutative rings with unit. $FGL(-)$ is represented by the Lazard ring L and $SI(-)$ is represented by the ring $LB = L \otimes Z[b_1, b_2, ...]$. In the grading introduced above, $|b_i| = 2i$.*

The statement of the representability of LB follows from that any matched pair (F, f, G) is determined by F and f, where $f(x) = x + \sum_{i\geq 0} f_i x^{i+1}$ is any power series, which is in 1-1 correspondence with ring homomorphisms $\theta : LB \to R$ with $\theta(b_i) = f_i$.

2.3 p-typical group law

Although it is not required for our proof of the main result about the Gysin homomorphism later, we define the p-typical group law and study some related properties for completeness.

Definition 2.3. *(p-typical law over a torsion-free $Z_{(p)}$-algebra) A formal group law over a torsion-free $Z_{(p)}$-algebra is p-typical if its logarithm has the form $\sum_{i\geq 0} l_i x^{p^i}$ with $l_0 = 1$.*

We can extend the concept of a p-typical formal group law to formal group laws over a general (not necessarily torsion-free) $Z_{(p)}$-algebra. Before doing that, we need some more definitions.

Definition 2.4. *Let F be a formal group law over R. If x and y are elements in an R-algebra A which also contains the power series $F(x, y)$, let*

$$x +_F y = F(x, y). \tag{13}$$

This notation may be iterated, e.g., $x +_F y +_F z = F(F(x, y), z)$. Similarly, $x -_F y = F(x, i(y))$, where $i(y)$ is the additive inverse characterised by $F(y, i(y)) = 0$. For nonnegative integers n, $[n]F(x) = F(x, [n-1]F(x))$ with $[0]F(x) = 0$. (The subscript F will be omitted if possible to do so without ambiguity.) $\sum^F()$ will denote the formal sum of the indicated elements.

For a non-zero integer n, we can define $[\frac{1}{n}]F(x)$ by the relation

$$F([\frac{1}{n}]F(x), [n]F(x)) = x, \tag{14}$$

and in general we can define $[\frac{p}{q}]F(x)$ for a rational number $\frac{p}{q}$ by the relation that

$$[q]F([\frac{p}{q}]F(x)) = [p]F(x). \tag{15}$$

It is straightforward to see that the following theorem follows from the preceding definition.

Theorem 2.6. *If the formal group law F above is defined over a K-algebra R where K is a subring of Q, then for each $r \in K$ there is a unique power series $[r]F(x)$ such that*

- *if r is a nonnegative integer, $[r]F(x)$ is the power series defined above,*
- $[r1 + r2]F(x) = F([r_1]F(x), [r_2]F(x))$,
- $[r_1 r_2]F(x) = [r_1]F([r_2]F(x))$.

Now we suppose q is a natural number which is invertible in R. Let

$$f_q(x) = [\frac{1}{q}](\sum_{i=1}^{q} \zeta^i x), \tag{16}$$

where by $\sum$ we mean $\sum^F$ and we define $f_1(x) = x$.

If R is torsion-free and $log(x) = \sum_{i>=0} m_i x^{i+1}$, we have

$$\begin{aligned} log(f_q(x)) &= [\frac{1}{q}] \sum_{i=1}^{q} log(\zeta^i x) \\ &= [\frac{1}{q}] \sum_{i=1}^{q} \sum_{j\geq 0} m_j x^{j+1} \zeta^{i(j+1)} \\ &= [\frac{1}{q}] \sum_{j\geq 0} m_j x^{j+1} \sum_{i=1}^{q} \zeta^{i(j+1)}, \end{aligned} \tag{17}$$

where ζ is a q^{th}-root of unity, i.e. ζ has order q in the ring $R[\zeta]$. The expression $\sum_{i=1}^{q} \zeta^{i(j+1)}$ vanishes unless $(j+1)$ is divisible by q, in which case its value is q. Hence, we have

$$log(f_q(x)) = \sum_{j>0} m_{qj-1} x^{qj}. \tag{18}$$

If F is p-typical for $p \neq q$, this expression vanishes. This motiates us to give the following definition of a p-typical group law for a general $Z_{(p)}$-algebra.

Definition 2.5. *A formal group law F over a $Z_{(p)}$-algebra is p-typical if $f_q(x) = 0$ for all primes $q \neq p$.*

Note because $log(x)$ is an isomorphism, this is precisely the definition for a p-typical formal group law over a torsion-free $Z_{(p)}$-algebra. It is a natural question to ask under what condition a formal group law is isomorphic to a p-typical group law. This is expressed by Cartier's theorem.

Theorem 2.7. *(Cartier) Every formal group law over a $Z_{(p)}$-algebra is canonically strictly isomorphic to a p-typical one.*

Proof. Let F be the universal group law over the Lazard ring L in (2.2), then it suffices to construct a strict isomorphism $\phi(x) = \sum \phi_i x^i \in L \otimes Z_{(p)}[[x]]$ from the image of F over $L \otimes Z_{(p)}$ to a p-typical formal group law F'. This is because if we are given a formal group law G over a Z_p-algebra R, and $\theta : L \to R$ is a ring homomorphism carrying F to G, then let $f'(x) = \sum \theta(\phi_i) x^i$, we can define a formal group law G' over R defined by $G'(x, y) = G(f'(x), f'(y))$, which is p-typical because L' is p-typical, and G is strictly isomorphic to G' via f'.

We define $\phi(x)$ by its inverse

$$\phi^{-1}(x) = \sum_{q>0, p\neq q}^{F} [\mu(q)]_F (f_q(x)), \tag{19}$$

where f_q is as in (16) and each q is a natural number. The function μ is the Möbius function

$$\mu(n) = \begin{cases} 1, & \text{if } n \text{ is square free and has an even number of prime factors} \\ -1, & \text{if } n \text{ is square free and has an odd number of prime factors} \\ 0, & otherwise. \end{cases} \tag{20}$$

Note that $\phi(x)$ is a strict isomorphism, because from (**??**) we have

$$f_q(x) \equiv 0 \mod x^q \tag{21}$$

for each $q > 0$, and in particular $f_1(x) = x$. This shows $\phi(x)$ is a well defined strict isomorphism, and we want to show it induces a p-typical law F'. The formal group law F' extends to $L \otimes Q$ and by an abuse of notation we identify this extended formal group law with F' and in the rest of the proof use F' to refer to this extension. Since the Lazard ring L is torsion-free and hence the inclusion map $L \otimes Z_p \to L \otimes Q$ is injective, it suffices to show F' is p-typical.

Notice $mog(x) = log(\phi^{-1}(x))$ is the logarithm of the formal group law F' induced by $\phi(x)$, and we have

$$
\begin{aligned}
log(\phi^{-1}(x)) &= \sum_{p\nmid q} \mu(q) \sum_{j>0} m_{qj-1} x^{qj} \\
&= \sum_{n>0} \left(\sum_{p\nmid q, q|n} \mu(q) \right) m_{n-1} x^n \qquad (22) \\
&= \sum_{i\geq 0} m_{p^i-1} x^{p^i},
\end{aligned}
$$

where we use the identity

$$
\sum_{p\nmid q, q|n} \mu(q) = \begin{cases} 1, & \text{if } n = p^k, \\ 0, & \textit{otherwise}, \end{cases} \qquad (23)
$$

so the formal group law F' is p-typical.

□

Cartier's construction enables us to construct the universal p-typical formal group law.

Theorem 2.8. *(Universal p-typical formal group law) Let $V = Z_{(p)}[v_1, v_2, ...]$ with $|v_n| = 2(p^n - 1)$. Then there is a universal p-typical formal group law F defined over V; i.e., for any p-typical formal group law G over a commutative $Z_{(p)}$-algebra R, there is a unique ring homomorphism $\theta : V \to R$ such that $G(x, y) = \theta(F(x, y))$. Moreover the homomorphism from $L \otimes Z_{(p)}$ to V induced by the universal property of the Lazard ring L is surjective.*

There is a ring VT similar to LB that represents the set of p-typical matched pairs (F, f, G). Before constructing VT, first note that if $f(x)$ is a power series on a ring R with zero constant term, and F is a formal group law over R, then we can write

$$
f(x) = \sum_{i>0}^{F} a_i x^i, \qquad (24)
$$

in a unique way where the coefficients of a_i are found inductively and the monomial term of degree k is only contributed by the terms of degrees no more than k in the formal sum (so the formal sum makes sense). In particular, this applies to any isomorphism $f(x)$ of formal group laws, in which case a_0 is a unit.

Lemma 2.1. *Let F be a p-typical formal group law over a $Z_{(p)}$-algebra R. Let $f(x)$ be an isomorphism from F to a formal group law G. Then G is p-typical if and only if*

$$
f^{-1}(x) = \sum_{i\geq o}^{F} t_i x^{p^i} \qquad (25)
$$

for $t_i \in R$ with t_0 a unit in R.

Proof. For a prime $q \neq p$, let

$$
\begin{aligned}
h_q(x) &= [\frac{1}{q}]_G \left(\sum_{i=1}^{q}{}^{G} \zeta^i x \right) \\
F_q(x) &= [\frac{1}{q}]_F \left(\sum_{i=1}^{q}{}^{F} \zeta^i x \right) = 0,
\end{aligned} \qquad (26)
$$

We need to show $h_q(x) = 0$ if and only if $f(x)$ is as specified. From the relation $G(x, y) = f(F(f^{-1}(x), f^{-1}(y)))$, we deduce

$$
f^{-1}(h_q(x)) = [\frac{1}{q}]_F \left(\sum_{j=1}^{q}{}^{F} f^{-1}(\zeta^j x) \right). \qquad (27)
$$

Let

$$f^{-1}(x) = \sum_{i>0}^{F} c_i x^i \tag{28}$$

with c_0 a unit in R. We have

$$\begin{aligned} f^{-1}(h_q(x)) &= [\frac{1}{q}]\left(\sum_{i=1}^{\infty}\sum_{j=1}^{q} c_i \zeta^{ij} x^i\right) \\ &= [\frac{1}{q}]\left(\sum_{q\nmid i}\sum_{j} \zeta^j c_i x^i + \sum_{i} [q]_F c_{qi} x^{qi}\right) \\ &= \sum_{q\nmid i} F_q(c_i x^i) + \sum_{i>0} c_{qi} x^{qi} \\ &= \sum_{i>0} c_{qi} x^{qi}, \end{aligned} \tag{29}$$

where all summation $\sum$ means summation under the formal group law F. The expression vanishes if and only if $f(x)$ is of the specified form.

□

2.4 The formal group law of complex cobordism

We are about to give the main result that there exists a formal group law F with coefficients in $\Omega^*(pt)$ the complex cobordism group of a point. Quillen proved [7] that the formal group law F is isomorphic to the universal formal group law over the Lazard ring.

We refer the reader to [2] for definition and properties of characteristic classes for complex cobordism. The key technical result is the following theorem due to Dold.

Proposition 2.1. *Given an $n+1$ dimensional complex vector bundle $E \to X$, let $\mathbb{P}E \to X$ be the bundle of projective lines in E, then the group $\Omega^*(\mathbb{P}E)$ is generated by $1, \zeta ..., \zeta^{n-1}$ as an $\Omega^*(X)$-module, where $\zeta = c_1(\mathcal{O}_1)$ is the first Chern class of the tautological bundle, and the generator satisfies the relation*

$$\zeta^n - f^*c_1(E)\zeta^{n-1} + ... + (-1)^n f^*c_n(E) = 0. \tag{30}$$

In particular, if $X = pt$, then we have $\Omega^*(\mathbb{CP}^n) = \Omega^*(pt)[x]/(x^n)$, and taking $X = \mathbb{CP}^n$ give

$$\Omega^*(\mathbb{CP}^n \times \mathbb{CP}^n) = \Omega^*(pt)[x_1, x_2]/(x_1^n, x_2^n). \tag{31}$$

Let $pr_i : \mathbb{CP}^n \times \mathbb{CP}^n \to \mathbb{CP}^n$ be the projection map for $i = 1$ or 2, then by (31) we have that

$$c_1(pr_1^*(\mathcal{O}_1) \otimes pr_2^*(\mathcal{O}_1)) = \sum_{0\le i<n, 0\le j<n} a_{ij} x_1^i x_2^j, \tag{32}$$

with coefficients a_{ij} in $\Omega^*(pt)$. If we increase n, then consider the canonical inclusion map $\mathbb{CP}^n \to \mathbb{CP}^{n+1}$ where the induced pullback maps the tautological bundle of $\mathbb{CP}^{n+1}$ to that of $\mathbb{CP}^n$, by the naturality of Chern classes, we can see that the coefficients a_{ij} are independent of the choice of n for n sufficiently large. Hence as we let n tend to infinity and consider the commutativity and associativity of taking tensor products, we obtain a formal group law

$$F(x, y) = \sum_{i\ge 0, j\ge 0} a_{ij} x^i y^j. \tag{33}$$

3 The residue symbol

In this section, we follow Quillen's notes to give a formula (first appearing in Cartier's unpublished notes) for the residue symbol. The overal construction is quite involved, but the example calculation subsection illustrates a few examples where the residue symple can be calculated in a straightforward way reminiscent of the usual notion of residue. We first recall the definition of the trace homomorphism for finitely generated projective modules.

3.1 The trace homomorphsim

For a finitely generateled projective R-module P and any R-module N, we have $N \otimes P^* \cong Hom_R(P, N)$, where P^* is the dual module of P. This follows from the fact that P is a direct summand of a finitely generated free module for which the statement holds. We define the trace homomorphism $tr : End_R(P) \cong P \otimes P^* :\to R$ by

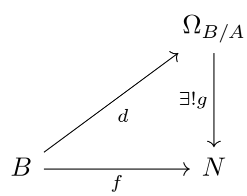
$$tr(\sum_{i}^{n}(x_i \otimes x_i^*))(p) = \sum_{i}^{n} x_i^*(p)x_i. \tag{34}$$

Concretely, given an endomorphism ϕ of P, we can pick a set of dual basis $\{x_i^*\}_{i=1}^n$ such that $\{x_i\}_{i=1}^n$ finitely generates P and $p = \sum_{i=1}^n x_i^*(p)x_i$ for each element p in P. Such a dual basis exists since we can pick a set of generators $\{x_i\}_{i=1}^n$ that induces a surjective map $R^n \to P$, and a section of this surjection gives arise to a dual basis after projection to each factor. In particular, fix a generating set of P, the trace of an endomorphism of P is the same as the matrix trace induced by this set of generators. One readily sees that the trace homomophism is commutative (same proof in linear algebra), i.e. $tr(AB) = tr(BA)$ for two endomorphisms.

3.2 Definition of the residue symbol

Let A be a commutative ring and let B be a commutative A-algebra. Fix $f \in B$ such that f is not a zero divisor and B/fB is a projective A-module. Let $\pi : B \to B/fB$ be the canonical projection map.

Let $\Omega_{B/A}$ be the B-module of Kähler differentials, which concretely is isomorphic to the B-module I/I^2, where $I \subset B \otimes_A B$ (viewed as B-module with multiplication on either side) is the B-module generated by elements of the form $b \otimes 1 - 1 \otimes b$ for $b \in B$. The B-module I is actually the kernel of the multiplication map $B \otimes_A B \to B$ and the B-module $\Omega_{B/A}$ has the following universal property [5]:



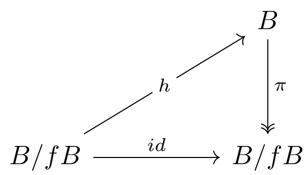
for any derivation f from B to a B-module N, i.e. f an additive map such that $f(ab) = af(b) + bf(a)$, there exists a unique map g such that the preceding diagram commutes, where the map $d : B \to \Omega_{B/A}$ is the universal derivation given by $d(b) = b \otimes 1 - 1 \otimes b$.

Let us now define the residue symbol $res : \Omega_{B/A} \to A$ which maps ω to $res \begin{bmatrix} \omega \\ f \end{bmatrix}$. We define $res \begin{bmatrix} \omega \\ f \end{bmatrix}$ first for $\omega = xdb$ and then extend it linearly to $\Omega_{B/A}$, so we consider fixed x and b in the following. Consider the following exact sequence

$$0 \longrightarrow B/fB \xrightarrow[(f)]{} B/f^2B \xrightarrow[\pi_{B/f^2B}]{} B/fB \longrightarrow 0,$$

where (f) is the map of multiplication by f and π_{B/f^2B} is the obvious projection map.

Since B/fB is projective, there exists a section $h : B/fB \to B$ such that $\pi h = id_{B/fB}$.



Let $[b,h] \in Hom_A(B/fB, B)$ denote the Lie bracket, i.e. $[b,h](u) = bh(u) - h(bu) \in B$. Notice

$$\pi([b,h](u)) = \pi(bh(u)) - \pi(h(bu)) = b(\pi(h(u))) - bu = 0, \tag{35}$$

and since f is not a zero divisor, there exists a unique element $a \in B$ such that $fa = [b,h](u)$. Viewing $(f)^{-1}$ as a function from fB to B, we have $(f)^{-1}[b,h](u) = a$ in B, so $(f)^{-1}[b,h]$ defines an element in $End_A B/fB$ after post-composition with the projection map π. Since B/fB is finitely generated as an A-module, there exists a trace map

$$tr : End_A(B/fB) \to A. \tag{36}$$

We define a derivation $D : B \to Hom_A(B/fB, A)$ by

$$D(b)([x]) = tr(\pi \circ (x(f)^{-1}[b,h])), \tag{37}$$

where $\circ$ denotes function compostion, $x \in B$ is a representative of $[x] \in B/fB$, and $End_A(B/fB)$ is viewed as a B-module with multiplication given by $b\phi(x) = \phi(bx)$, where $\phi \in End_A(B/fB)$ and $b \in B$ (similarly we can define a B-module structure on $Hom_A(N, C)$ for any A-module C and B-module N this way).

Lemma 3.1. *The equation* (37) *defines a derivation* D*.*

Proof. Firstly, notice that $[b,h] = (b) \circ h - bh$, where (b) is the function of multiplication by b, and bh comes from scalar multiplication in the B-module $Hom_A(B/fB, B)$. Therefore, we have

$$\begin{aligned} [b_1b_2, h] &= (b_1b_2) \circ h - b_1b_2h = (b_1) \circ (b_2) \circ h - b_1b_2h, \\ [b_2, h] \circ (b_1) &= (b_2) \circ h \circ (b_1) - b_1b_2h, \\ (b_2) \circ [b_1, h] &= (b_2) \circ (b_1) \circ h - (b_2) \circ h \circ (b_1), \end{aligned} \tag{38}$$

so $[b_1b_2, h] = [b_2, h] \circ (b_1) + (b_2) \circ [b_1, h]$. Therefore,

$$\begin{aligned} (f)^{-1}[b_1b_2, h] &= (f)^{-1} \circ ([b_2, h] \circ (b_1) + (b_2) \circ [b_1, h]) \\ &= (f)^{-1} \circ ([b_2, h] \circ (b_1)) + (f)^{-1} \circ ((b_2) \circ [b_1, h]) \\ &= (f)^{-1} \circ [b_2, h] \circ (b_1) + (b_2) \circ (f)^{-1} \circ [b_1, h] \end{aligned} \tag{39}$$

,

where the last step follows since $(f)^{-1} \circ (b) = (b) \circ (f)^{-1}$ as a function from fB to B. Composing with the projection map, we have

$$\begin{aligned} \pi \circ (f)^{-1}[b_1b_2, h] &= \pi \circ (f)^{-1} \circ [b_2, h] \circ (b_1) + \pi \circ (b_2) \circ (f)^{-1} \circ [b_1, h] \\ &= b_1(\pi 3 \circ (f)^{-1} \circ [b_2, h]) + ([b_2]) \circ (\pi \circ (f)^{-1} \circ [b_1, h]), \end{aligned} \tag{40}$$

where $[b_2]$ is the image of b_2 in B/fB.

Thus we have

$$\begin{aligned} D(b_1b_2)(x) &= tr\left(\pi \circ (x(f)^{-1}[b_1b_2, h])\right) \\ &= tr\left(\pi \circ ((xb_1)(f)^{-1}([b_2, h]))\right) + tr\left(([b_2]) \circ (\pi \circ (f)^{-1} \circ [b_1, h] \circ (x))\right) \\ &= (b_1D(b_2))(x) + tr\left((\pi \circ (f)^{-1} \circ [b_1, h] \circ (xb_2))\right) \\ &= (b_1D(b_2))(x) + (b_2D(b_1))(x), \end{aligned} \tag{41}$$

where the second last step follows from the fact that the trace operator is commutative. $\square$

The derivation D gives rise to a unique B-homomorphism $\theta : \Omega_{B/A} \to Hom_A(B, A)$ such that

$$\begin{array}{ccc} & & \Omega_{B/A} \\ & \overset{d}{\nearrow} & \Big\downarrow \theta \\ B & \xrightarrow[D]{} & Hom_A(B, A) \end{array}$$

commutes. By post-composing with the evaluation map at 1, we obtain the map

$$res(\begin{bmatrix} \cdot \\ f \end{bmatrix}) : \Omega_{B/A} \to A \tag{42}$$

characterized by the formula

$$res(\begin{bmatrix} xdb \\ f \end{bmatrix}) = tr\left(\pi(x(f)^{-1}[b, h])\right). \tag{43}$$

Recall the earlier choice of a section $h : B/fB \to B$.

Lemma 3.2. *The residue symbol for f is independent of the choice of h.*

Proof. Suppose we choose a different h', then $h - h' = (f) \circ \phi$ for some $\phi \in Hom_A(B/fB, B)$, and

$$\begin{aligned} tr\left(\pi(x(f)^{-1}[b, h])\right) - tr\left(\pi(x(f)^{-1}[b, h'])\right) &= tr\left(\pi(x(f)^{-1}[b, (f) \circ \phi])\right) \\ &= tr\left(([f]) \circ \pi(x(f)^{-1}[b, \phi])\right) \\ &= 0, \end{aligned} \tag{44}$$

so the residue symbol is well defined. $\square$

3.3 Scaling invariance

Before presenting concrete example calculations, we prove that the residue symbol is invariant under scaling in the following sense.

Proposition 3.1. *Let the residue symbol be defined as previously, we have*

$$res\begin{bmatrix} a\omega \\ af \end{bmatrix} = res\begin{bmatrix} \omega \\ f \end{bmatrix}, \tag{45}$$

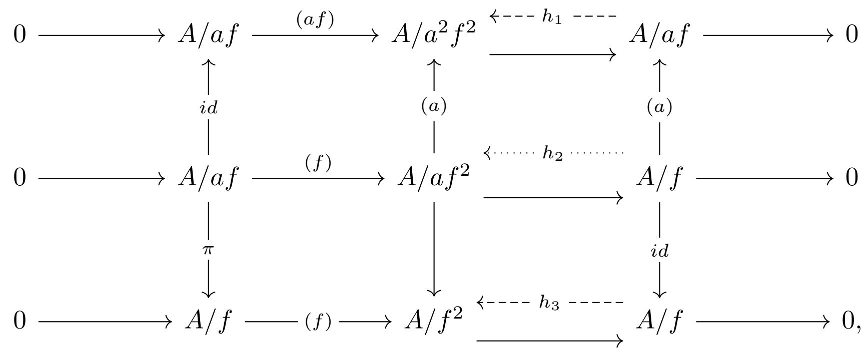
where a and f are non-zero-divisors in an A-algebra B such that B/aB and B/afB are finitely generated projective A-modules.

Proof. By linearity, it suffices to assume $\omega = xdy$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A/af^2 & \xrightarrow{(a)} & A/a^2f^2 & \longrightarrow & A/a & \longrightarrow & 0 \\ & & \Big\downarrow \; \Big\uparrow h_2 & & \Big\downarrow \; \Big\uparrow h_1 & & \Big\downarrow id & & \\ 0 & \longrightarrow & A/f & \xrightarrow{(a)} & A/af & \longrightarrow & A/a & \longrightarrow & 0, \end{array}$$

where (a) is the map induced by multiplication by a, and the unlablelled maps are the obvious projection maps. Since A/f and A/af are projective A-modules, the identity map of A/a induces a section h_1, and h_1 induces another section h_2.

The key commutative diagram is the following



where π and the unlabelled maps are natural projection maps. We have

$$\begin{aligned} res\begin{bmatrix} a\omega \\ af \end{bmatrix} &= tr_{A/af}(ax(af)^{-1}[y,h_1]) \\ &= tr_{aA/af}(ax(af)^{-1}[y,h_1]) \\ &= tr_{A/f}(x(af)^{-1}[y,h_1]\circ(a)) \\ &= tr_{A/f}(x(af)^{-1}[y,h_3]) \\ &= res\begin{bmatrix} \omega \\ f \end{bmatrix}, \end{aligned} \tag{46}$$

where the first and last equality signs follow by definition, the second equality sign follows because the image of the endomorphism is in aA/af, and the last equality sign follows from diagram chasing.

□

3.4 Example calculation

Suppose that $B = A[[Z]]$ the ring of formal power series with coefficients in a commutative ring A, and $f \in B$ has the following form $f(Z) = Z^n - c_1 Z^{n-1} + \ldots + (-1)^n c_n$, where the $\{c_i\}$s are nilpotents elements of A. Let $\pi : B \to B/fB$ denote the canonical projection map. We have

$$B/fB = \bigoplus_{i=0}^{n-1} A\alpha^i, \tag{47}$$

as an A-algebra, and the generator α satisfies

$$\alpha^n = c_{n-1}\alpha^{n-1} + \ldots + (-1)^{n+1}c_n, \tag{48}$$

which is a sum of nilpotent elements, so α is nilpotent. Let $h : B/fB \to B = A[[Z]]$ be the section that lifts the identity map given by $h(\alpha^j) = Z^j$ where $0 \le j < n$. We have $(Z)h(\alpha^j) = Z^{j+1}$ and

$$Zh(\alpha^j) = h(\alpha^{j+1}) = \begin{cases} Z^{j+1}, & 0 \le j < n-1 \\ c_{n-1}Z^{n-1} + \ldots + (-1)^n c_n, & j = n-1 \end{cases}, \tag{49}$$

hence

$$[Z,h](\alpha^j) = \begin{cases} 0 & 0 \le j < n-1 \\ Z^n - c_{n-1}Z^{n-1} + \ldots + (-1)^n c_n, & j = n-1 \end{cases}, \tag{50}$$

and

$$(f)^{-1}[Z,h](\alpha^j) = \begin{cases} 0 & 0 \le j < n-1 \\ 1 & j = n-1 \end{cases}, \tag{51}$$

so

$$\begin{aligned}
res\begin{bmatrix} Z^j dZ \\ f(Z) \end{bmatrix} &= res\begin{bmatrix} Z^j dZ \\ Z^n - c_{n-1}Z^{n-1} + \ldots + (-1)^n c_n \end{bmatrix} \\
&= tr\left(\pi(Z^j (f)^{-1}[Z,h](\alpha^j))\right) \\
&= \begin{cases} 0 & 0 \leq j < n-1 \\ 1 & j = n-1 \end{cases}
\end{aligned} \tag{52}$$

Moreover, since α is nilpotent (as linear combination of nilpotent elements), we have that there exists a natural number N such that $Z^n - c_{n-1}Z^{n-1} + \ldots + (-1)^n c_n$ divides Z^N. For an element $P(Z) \in B$, let $P_m(Z)$ denote the truncated polynomial of $P(Z)$ up to the term containing the monomial Z^m. By (3.1) and the fact that $res\left[\begin{smallmatrix} - \\ 1 \end{smallmatrix}\right]$ is the constant 0 map, which implies $res\left[\begin{smallmatrix} f(Z)dZ \\ f(Z) \end{smallmatrix}\right] = 0$ we have that $\pi(P_m(Z)) = \pi(P_N(Z))$ if $m \geq N$, and

$$res\begin{bmatrix} P(Z)dZ \\ f(Z) \end{bmatrix} = res\begin{bmatrix} P_m(Z)dZ \\ f(Z) \end{bmatrix} = res\begin{bmatrix} P_N(Z)dZ \\ f(Z) \end{bmatrix} = res\begin{bmatrix} R(Z)dZ \\ f(Z) \end{bmatrix}, \tag{53}$$

where $R(Z)$ is the remainder polynomial of degree at most $n-1$ by dividing $P_N(Z)$ by $f(Z)$ using the division algorithm.

In particular, we have the following useful identities.

Proposition 3.2. *With $A[[Z]]$ in the previous section, we have*

$$res\begin{bmatrix} \sum_{i=0}^{\infty} a_i Z^i dZ \\ Z^n \end{bmatrix} = a_{i-1} \tag{54}$$

$$res\begin{bmatrix} P(Z)dZ \\ Z-a \end{bmatrix} = P(a), \tag{55}$$

for $P(Z) \in A[[Z]]$.

Proof. If $f(Z) = Z^n$, by (52) we have that $res\left[\begin{smallmatrix} P(Z)dZ \\ f(Z) \end{smallmatrix}\right]$ is the coefficient of Z^{n-1} in $P(Z)$. If $f(Z) = Z - c$, then $R(Z)$ is just the $P_N(Z)$ evaluated at c, so

$$res\begin{bmatrix} P(Z)dZ \\ Z-c \end{bmatrix} = res\begin{bmatrix} P_N(Z)dZ \\ Z-c \end{bmatrix} = P_N(c) = P(c), \tag{56}$$

where $P(c)$ is well defined by the previous discussion of the nilpotence of c and that $P_m(Z) = P_N(Z)$ for $m \geq N$. ☐

The residue symbol $res\left[\begin{smallmatrix} db \\ f \end{smallmatrix}\right]$ defined on the A-module $A[[Z]]$ is also compatible with the change of base in the following sense.

Proposition 3.3. *Let $f \in A[[Z]] = Z^n - c_{n-1}Z^{n-1} + \ldots + (-1)^n c_n$, where the the coefficients c_i are nilpotent and $n > 0$. Given a ring homomorphism $\phi : A \to A'$, so that $A[[Z]]/f$ and $A'[[Z]]/\phi(f)$ are free modules of the same dimension over the rings A and A', respectively, we have*

$$\phi\left(res\begin{bmatrix} b \\ f \end{bmatrix}\right) = res\begin{bmatrix} \phi(b) \\ \phi(f) \end{bmatrix} \tag{57}$$

.

Proof. By linearity of the residue symbol, it suffices to prove the result for $b = adx$. Let $h : A[[Z]]/f \to A[[Z]]$ be a section of the identity map on $A[[Z]]/f$ such that $h((Z^j)) = Z^j$, where (Z^j) denotes the equivalence class of Z^j and $0 < j < n$. Similarly define $h' : A'[[Z]]/f \to A'[[Z]]$ as a section of the identity map on $A'[[Z]]/f$, then

$$
\begin{aligned}
res(\begin{bmatrix} \phi(x)d\phi(b) \\ \phi(f) \end{bmatrix}) &= tr\left(\phi(x)(\phi(f))^{-1}[\phi(b), h']\right) \\
&= \phi(tr\left(x(f)^{-1}[b,h]\right)) \\
&= \phi(res\begin{bmatrix} xdb \\ f \end{bmatrix})
\end{aligned}
\tag{58}
$$

□

3.5 The splitting principle

We now state a key result of the splitting principle that will be useful for calculating the Gysin homomorphism of complex cobordism.

Proposition 3.4. *Let $\pi : E \to X$ be a complex vector bundle, then there exists a manifold P and such that the following commutative diagram is cartesian*

$$
\begin{array}{ccc}
E' & \longrightarrow & E \\
\big\downarrow & & \big\downarrow \pi \\
P & \xrightarrow{\quad f \quad} & X,
\end{array}
$$

E' is a direct sum of line bundles, and the induced map $f^* : \Omega^*(X) \to \Omega^*(P)$ is injective.

Proof. Consider the $f : \mathbb{P}E^* \to X$ the projective bundle of lines in E, then the pullback bundle $E' = f^*(E)$ contains the line bundle $\mathcal{O}_1$ of $\mathbb{P}E^*$ as a direct summand. The map f^* is injective by (2.1). If $dim(E)$ is 1 or 2, then we are done, otherwise repeat the process on $E' \to P$ to split the remaining summand into a direct sum of line bundles. □

3.6 Calculating the Gysin homomorphism

Let E be a complex vector bundle over X of dimension n. Let $f : \mathbb{P}E^* \to X$ be the associated projective line bundle. Recall that $\Omega^*(\mathbb{P}E^*)$ is a free module over $\Omega^*(X)$ with generators $1, \zeta, ..., \zeta^{n-1}$, where $\zeta = c_1(\mathcal{O}_1)$ is the first Chern class of the canonical line bundle and ζ satisfies the following relation

$$
\zeta^n - f^*c_1(E)\zeta^{n-1} + ... + (-1)^n f^*c_n(E) = 0. \tag{59}
$$

Let $F(x, y) \in \Omega^*(pt)[[x, y]]$ be the formal group law associated with taking the first Chern class of the tensor product of line bundles, and let $I(x) \in \Omega^*[[x]]$ be the inverse such that $F(x, I(x)) = 0$. The invariant differential ω is defined by

$$
\omega = \frac{dZ}{\frac{\partial}{\partial y}F(Z, 0)}, \tag{60}
$$

which is equal to $\frac{\partial}{\partial z}log(Z)$ if the logarithm for the formal group law exists (e.g. after tensoring with $\mathbb{Q}$).

Also observe that $\Omega^*(\mathbb{P}E^*)[[Z]]$ has the structure of a free $\Omega^*(X)[[Z]]$-module with generators $1, \zeta, ..., \zeta^{n-1}$ and multiplication is induced by multiplication of the $\Omega^*(X)$-module $\Omega^*(\mathbb{P}E^*)$ for each monomial.

The main theorem is the following due to Quillen.

Theorem 3.1. *If $a(Z) \in \Omega^*(X)[Z]$, then*

$$
f_*(a(\zeta)) = res\begin{bmatrix} a(Z)\omega \\ Norm(F(Z, I\zeta)) \end{bmatrix}, \tag{61}
$$

where $F(Z, I\zeta) \in \Omega^*(X)[[Z]]$ is well defined since ζ is nilpotent, and multiplication by $F(Z, I\zeta)$ defines an endomorphism of the free $\Omega^*(X)[[Z]]$-module $\Omega^*(\mathbb{P}E^*)[[Z]]$ so that $Norm(F(Z, I\zeta))$ is the determinant of this endomorphism represented by a matrix on a set of generators.

Before proving the main theorem, we would need to first prove a technical lemma.

Consider $n \times n$ matrices ($n \geq 2$) of the forms

$$A_n(Z, a) = \begin{bmatrix} Z & -1 & & & \\ & Z & -1 & & \\ & & Z & -1 & \\ & & & \ddots & \\ a & & & & Z \end{bmatrix} \tag{62}$$

$$B_n(Z, a) = \begin{bmatrix} 0 & -1 & & & \\ & Z & -1 & & \\ & & Z & -1 & \\ & & & \ddots & \\ a & & & & Z \end{bmatrix}, \tag{63}$$

where the unfilled entries are 0. The endomorphism induced by multiplication by $Z - \zeta$ can be represented by the matrix $A_n(Z, g(Z))$ for the basis $1, \zeta, ...\zeta^{n-1}$, where

$$g(Z) = -f^*c_1(E)Z^{n-1} + ... + (-1)^n f^*c_n(E). \tag{64}$$

Lemma 3.3.

$$Norm(Z - \zeta) = Z^n - f^*c_1(E)Z^{n-1} + ... + (-1)^n f^*c_n(E) \tag{65}$$

Proof. It suffices to give a general formula for computing the determinants of $A_n(Z, a)$ and $B_n(Z, a)$.

We prove that $det(A_n(Z, a)) = Z^n + a$ and $det(B_n(Z, a)) = a$ by induction. The base case is a straightforward calculation for $n = 2$. For the inductive case, notice

$$\begin{aligned} det(A_{n+1}(Z, a)) &= Zdet(A_n(Z, 0)) + det(B_n(Z, a)) \\ &= Z^{n+1} + a, \end{aligned} \tag{66}$$

and

$$det(B_{n+1}(Z, a)) = det(B_n(Z, a)) = a. \tag{67}$$

This finishes the proof by induction, so $Norm(Z - \zeta) = Z^n - f^*c_1(E)Z^{n-1} + ... + (-1)^n f^*c_n(E)$. □

We now show that the residue symbol in (3.1) is well defined.

Lemma 3.4. *The res $\left[\frac{a(Z)\omega}{Norm(F(Z,I\zeta))}\right]$ residue symbol is well defined.*

Proof. Let

$$F(X, Y) = X + Y + XYG(X, Y), \tag{68}$$

and $Z = F(X, Y)$, or $Y = F(Z, IX)$, we obtain the identity

$$Z - X = F(Z, IX)(1 + XG(X, F(Z, IX))), \tag{69}$$

so

$$Z - \zeta = F(Z, I\zeta)(1 + \zeta G(\zeta, F(Z, I\zeta))) \in \Omega^*(\mathbb{P}E^*), \tag{70}$$

where since ζ is nilpotent, every term involving ζ is well defined and $1+\zeta G(\zeta, F(Z, I\zeta))$ is a unit. Therefore,

$$\begin{aligned} Norm(F(Z,I\zeta)) &= (unit) Norm(Z-\zeta) \\ &= (unit)(Z^n - f^* c_1(E) Z^{n-1} + ... + (-1)^n f^* c_n(E)), \end{aligned} \tag{71}$$

where the last step follows from (3.3). ☐

Therefore, $Norm(F(Z,I\zeta)) = Z^n - f^*c_1(E)Z^{n-1} + ... + (-1)$ is a non-zero-divisor in $\Omega^*(X)[[Z]]$ and $\Omega^*(X)[[Z]]/Norm(F(Z,I\zeta))$ is a finitely generated free module over $\Omega^*(X)$, so the residue symbol is well defined.

By the fact the pullback map of complex cobordism is a ring homomorphism and (57), both sides of the equation (3.1) are compatible with base change, by the splitting principle (3.4), we may assume that $E = L_1 + ... + L_n$ is a direct sum of line bundles. Letting $x_i = c_1(\mathcal{O}_i)$, we have

$$Norm(F(Z,I\zeta)) = \prod_{i=1}^{n} F(Z, Ix_i). \tag{72}$$

We now prove the formula in (3.1) by induction on n.

Lemma 3.5. *The formula* (3.1) *is true for* $n=1$.

Proof. Suppose $n=1$, so $\mathbb{P}E^* \cong X$ and $f_*(a(\zeta)) = a(\zeta)$. Since x_i is nilpotent, by (70) and (3.1) we have

$$\begin{aligned} res\begin{bmatrix} a(Z)\omega \\ F(Z,Ix_1) \end{bmatrix} &= res\begin{bmatrix} (1+x_1G(Z,F(Z,Ix_1)))a(Z)\omega \\ Z-x_1 \end{bmatrix} \\ &= \frac{(1+x_1G(x_1,0))a(x_i)}{\frac{\partial}{\partial y}F(x_1,0)} \\ &= a(x_1), \end{aligned} \tag{73}$$

where the first step follows from (70), the second step follows from (56), and the last follows by differentiating (68) with respect to Y. This finishes the proof for the base case. ☐

For the inductive step, suppose $n>1$ and $F = L_1 \oplus ... \oplus L_{n-1}$. Let

$$\mathbb{P}L_n^* \xrightarrow{\;i\;} \mathbb{P}E^* \xleftarrow{\;j\;} \mathbb{P}F^*$$

be the inclusion maps. Let $\iota : \mathbb{P}E^* \to \mathcal{O}_1 \otimes f^*F^*$ be the zero section and let ι' be a homotopy of ι. We need the following key lemma.

Lemma 3.6. *With the relevant bundles defined above, we have the following commutative diagrams*

$$\begin{array}{ccc} \mathbb{P}F^* & \longrightarrow & \mathbb{P}E^* \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \iota'} \\ \mathbb{P}E^* & \xrightarrow{\;\iota\;} & \mathcal{O}_1 \otimes f^*L_n^* \end{array} \tag{74}$$

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}E^* \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \iota'} \\ \mathbb{P}E^* & \xrightarrow{\;\iota\;} & \mathcal{O}_1 \otimes f^*F^*. \end{array} \tag{75}$$

Proof. To prove the theorem, it suffices construct the homotopies directly. The key observation is that for $\alpha \in \mathbb{P}E^*$, we can pick a neighbourhood U_α of α so that in local coordinates an element of U_α can be written as $(x_1, ..., x_m, y_1, ..., y_n)$ under the equivalence relation $(x_1, ..., x_m, y_1, ..., y_n) \sim (x_1, ..., x_m, cy_1, ..., cy_n)$, where $c \in \mathbb{C}$, and the coordinates x_j and y_i come from some trivialization coordinate charts on the manifold X and L_i^*, respectively. We define a function $f_s^i : \mathbb{P}E^*\big|_{U_\alpha} \to \mathbb{R}$ by

$$f_s^i(x_1, ..., x_m, y_1, ..., y_n) = s\frac{y_i\bar{y_i}}{\sum_{i=1}^n y_n\bar{y_n}}, \tag{76}$$

where $s \in [0, 1]$. Then, let ϕ_i be the section of $\mathcal{O}_1 \otimes f^*L_i^* \to \mathbb{P}E^*$ defined fiberwise where the i^{th} coordinate does not vanish, that maps each point to a constant unit vector above, then $f_s^i\phi_s$ is a section of $\mathcal{O}_1 \otimes f^*L_i^*$ well defined over U_α. In particular, if $i = n$, then the zero locus is $\mathbb{P}F^*\big|_{U_\alpha}$.

Since a manifold is paracompact, we can form the sections $f_s^i\phi_i$ for a locally finite cover to form a section

$$f_s^i\phi_i : \mathbb{P}E^* \to \mathcal{O}_1 \otimes f^*L_i^*. \tag{77}$$

Hence we can take $\iota' = f_1^n\phi_1$ in (74) and $\iota' = \oplus_{i=1}^{n-1} f_1^i\phi_i$ in (75), and the result follows. ☐

This completes the proof of (3.1), which follows from (3.5) and (3.6).

Corollary 3.1. *With*

$$\mathbb{P}L_n^* \xrightarrow{\ i\ } \mathbb{P}E^* \xleftarrow{\ j\ } \mathbb{P}F^*$$

defined as before, we have

$$i_*(1) = c_{n_1}(\mathcal{O}_1 \otimes f^*F) = \prod_{j=1}^n F(\zeta, Ix_j) \tag{78}$$

and

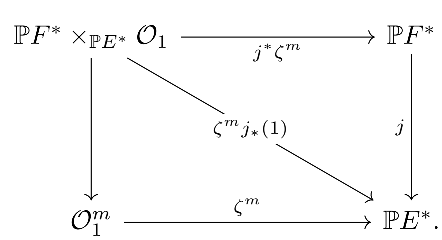
$$j_*(1) = c_1(\mathcal{O}_1 \otimes f^*L_n) = F(\zeta, Ix_n). \tag{79}$$

Proof. This follows from (3.6) and one can check stable normal complex structures are compatible. ☐

Now we continue to prove the inductive hypothesis. Firstly, we show the formula holds if $F(Z, Ix_n)$ divides $a(Z)$, i.e. $a(Z) = \alpha(Z)F(Z, Ix_n)$. Since

$$\begin{aligned} f_*(\alpha(\zeta)F(\zeta, Ix_n)) &= f_*(\alpha(\zeta)j_*(1)) \\ &= f_*j_*(\alpha(j^*\zeta)) \\ &= g_*\alpha(\zeta') \end{aligned} \tag{80}$$

where $g : \mathbb{P}F^* \to X$ is the canonical projection map and $\zeta' = c_1(\mathcal{O}_1)$ on $\mathbb{P}F^*$. The second step follows from the linearity of f_* and j_* and the below commutative diagram (so in particular, if $\alpha = \zeta^m$, then $\alpha(\zeta)j_*(1) = j_*(j^*(\zeta^m)) = j_*((j^*\zeta)^m)$ holds)



We also have

$$res\left[\frac{\alpha(Z)F(Z, Ix_n)\omega}{\prod_{j=1}^n F(Z, Ix_j)}\right] = res\left[\frac{\alpha(Z)\omega}{\prod_{j=1}^{n-1} F(Z, Ix_j)}\right], \tag{81}$$

which implies

$$\begin{aligned} res\left[\frac{\alpha(Z)F(Z,Ix_n)\omega}{\prod_{j=1}^{n} F(Z,Ix_j)}\right] &= res\left[\frac{\alpha(Z)\omega}{\prod_{j=1}^{n-1} F(Z,Ix_j)}\right] \\ &= g_*\alpha(\zeta') \\ &= f_*(\alpha(\zeta)F(\zeta,Ix_n)) \end{aligned} \tag{82}$$

by the inductive hypothesis and (80). Since $Z - x_n = F(Z, Ix_n)(unit)$, the formula holds for $a(Z) = \alpha(Z)(Z - x_n)$. Similarly, we can prove the formula holds for $a(Z) = \alpha(Z)\prod_{j<n}(Z - x_j)$.

By the division algorrithm,

$$\prod_{j<n}(Z - x_j) = g(Z)(Z - x_n) + \prod_{j<n}(x_n - x_j), \tag{83}$$

so the formula holds if $a(Z)$ is a multiple of $\prod_{j<n}(x_n - x_j)$, and hence

$$\prod_{j<n}(x_n - x_j)(f_*(\zeta) - res\left[\frac{a(Z)\omega}{\prod_{j=1}^{n} F(Z,Ix_j)}\right]) = 0 \tag{84}$$

To finish the proof, we use the following key observation.

Lemma 3.7. *There exists a sufficiently large natural number N and maps $h_i : X \to \mathbb{CP}^N$ such that $h_i^*(\mathcal{O}_1) = L_i$. Consider the bundle*

$$E' = L'_1 \oplus ... \oplus L'_n \to X' = \mathbb{CP}^N \times ... \times \mathbb{CP}^N, \tag{85}$$

where X' is n copies of $\mathbb{CP}^N$ and E' is the direct sum of line bundles L'_i that is $\mathcal{O}_1$ when restricted to the i^{th} factor, and let $h : X \to X'$ be the map that agrees with h_i on the i^{th} factor. Let ζ' be canonical line bundle of the projective bundle $f' : \mathbb{E}' \to X'$, then $\zeta = h'^*(\zeta')$ where h' is the bundle homomorphism induced by h, and

$$f_*(\zeta) = h^*(f'_*(\zeta')) \tag{86}$$

$$res\left[\frac{a(Z)\omega}{\prod_{j=1}^{n} F(Z,Ix_j)}\right] = h^*(res\left[\frac{a(Z)\omega}{\prod_{j=1}^{n} F(Z,Ix'_j)}\right]), \tag{87}$$

where $x'_j = c_1(L'_j)$ and $h^*(x'_j) = x_j$.

Proof. The identity (86) follows from the fact that the Gysin homormophism commutes with pullback, so $f_*h'^* = h^*f'_*$ where

$$\begin{array}{ccc} E & \xrightarrow{\ h'\ } & E' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow{\ h\ } & X'. \end{array}$$

The identity (87) follows from the change of base formula (57) applied to $h^* : \Omega^*(X') \to \Omega^*(X)$.

□

Consider the identity

$$\prod_{j<n}(x'_j - x'_n)(f'_*(a(\zeta)) - res\left[\frac{a(Z)\omega}{\prod_{j=1}^{n} F(Z,Ix'_j)}\right]) = 0 \in \Omega^*(X'). \tag{88}$$

Since $\Omega^*(X') \cong \Omega^*(pt)[x'_1, x'_2...x'_n]/(x_1'^{N+1}, ..., x_n'^{N+1})$, if we let N tend to infinity, we have that

$$\prod_{j<n}(x'_j - x'_n)u = 0 \tag{89}$$

in $\Omega^*(pt)[[x'_1, x'_2...x'_n]]$, where u is such that its pullback is equal to

$$f'_*(a(\zeta)) - res\left[\frac{a(Z)\omega}{\prod_{j=1}^n F(Z, Ix'_j)}\right] \tag{90}$$

along the canonical inclusion of $\Omega^*(pt)[x'_1, x'_2...x'_n]/(x'^{N+1})$ (indeed, this pullback is the truncated polynomial of terms containing no power of degree higher than $n+1$). We have $\prod_{j<n}(x'_j - x'_n)u = 0$ since (88) implies all its coefficients are 0.

The elements $x'_j - x'_n$ are non-zero-divisors in $\Omega^*(pt)[[x'_1, x'_2...x'_n]]$, so

$$f'_*(a(\zeta)) - res\left[\frac{a(Z)\omega}{\prod_{j=1}^n F(Z, Ix'_j)}\right] = 0, \tag{91}$$

this implies $f_*(a(\zeta)) - res\left[\frac{a(Z)\omega}{\prod_{j=1}^n F(Z,Ix_j)}\right]$ is 0 since by (88) it is the pullback along h of $f'_*(a(\zeta')) - res\left[\frac{a(Z)\omega}{\prod_{j=1}^n F(Z,Ix'_j)}\right] = 0$.

3.7 Myshenko's formula

A direct application of (3.1) is the following result due to Myshenko.

Proposition 3.5. *The invariant differential associated with the formal group law over $\Omega^*(pt)[[x_1, x_2]]$.*

$$\omega = \sum_{n=0}^{\infty} \mathbb{CP}^n Z^n dZ. \tag{92}$$

Proof. Applying (3.1) to the trivial bundle $\pi : \mathbb{C}^{n+1} \to pt$ and $a(Z) = 1$, we have

$$\begin{aligned} \pi_*(1) &= res\left[\frac{\omega}{\prod_{i=0}^{n+1} F(Z, Ix_i)}\right] \\ &= res\left[\frac{\omega}{\prod_{i=0}^{n+1} F(Z, 0)}\right] \\ &= res\left[\frac{\omega}{Z^{n+1}}\right] \end{aligned}, \tag{93}$$

where the second step follows because the Chern classes are trivial. Hence $\omega = \sum_{n=0}^{\infty} \mathbb{CP}^n Z^n$ follows by (3.2).

□

Over $\Omega^*(pt)[[x_1, x_2] \otimes \mathbb{Q}$ there exists a unqiue logarithm log such that

$$log(F(x, y)) = log(x) + log(y), \tag{94}$$

$$log(0) = 0, \tag{95}$$

and

$$dlog(Z) = \omega. \tag{96}$$

Hence we have Myshenkos' formula

$$log(Z) = \sum_{n=0}^{\infty} \frac{\mathbb{CP}^n Z^{n+1}}{n+1}. \tag{97}$$

3.8 Complex cobordism ring as the Lazard ring

In this section, we show that the complex cobordism ring $\Omega^*(pt)$ is isomorphic to the Lazard ring L, which is the universal ring for formal group laws.

Theorem 3.2. *Let $h : L \to \Omega^*(pt)$ be the map induced by the formal group law F on $\Omega^*(pt)$ defined by (33) that corresponds to taking the Chern class of the tensor product of line bundles. Then h is an isomorphism.*

Proof. Notice first that F induces a formal group law over the Q-algebra $\Omega^*(pt) \otimes Q$, which is isomorphic to $Q[\mathbb{CP}^1, \mathbb{CP}^2, ...]$ by results about rational homotopy, where $\mathbb{CP}^i$ represents the complex cobordism class in $\Omega^*(pt)$ as the i-dimensinal projective space. Moreover, by (2.3), we have that $L \otimes Q = Q[m_1, m_2, ...]$, where $f(x) = x + \sum_{i>0} m_i x^{i+1}$ is the logarithm for the formal group law induced by the universal formal group law over $L \otimes Q$. Then

$$h' : L \otimes Q \to \Omega^*(pt) \otimes Q \tag{98}$$

is an isomorphism, because it maps the generators to the generators.

By Lazard's results on the structure of L, L is torsion free, so h is injective. So it suffices to show that h is surjective.

By a result of Milnor, the ring $\Omega^*(pt)$ is generated by elements of the form $[M]$, where $M \in \mathbb{CP}^{r_1} \times ... \times \mathbb{CP}^{r_n}$ is a non-singular hypersurface of degree $k_{r_1}, ..., k_{r_n}$. In particular, let $\pi : \mathbb{CP}^{r_1} \times ... \times \mathbb{CP}^{r_n} \to pt$ be the projection map, we have

$$M = \pi_*(c_1(L_1^{k_1} \otimes ... \otimes L_n^{k_n})), \tag{99}$$

where L^{k_i} is the direct summand of k_i copies of the line bundle L_i which is the pullback of the canonical bundle on the i^{th} factor.

The equation for M can be rewritten in the following form using the formal group law

$$\begin{aligned} M &= \pi_* \left(\sum_{i_1,...,i_n} \pi^*(a_{i_1...i_n}) z_1^{i_1} ... z_n^{i_n} \right) \\ &= \sum_{i_1,...,i_n} \pi_*(\pi^*(a_{i_1...i_n})) \pi_*(z_1^{i_1} ... z_n^{i_n}) \\ &= \sum_{i_1,...,i_n} \pi_*(\pi^*(a_{i_1...i_n})) \mathbb{CP}^1 ... \mathbb{CP}^n. \end{aligned} \tag{100}$$

Notice that if α is stably complex manifold representing $a_{i_1...i_n}$, then $\pi^*(a_{i_1...i_n})$ is represented by $\mathbb{CP}^{r_1} \times ... \times \mathbb{CP}^{r_n} \times \alpha$ with the normal complex structure in $\mathbb{CP}^{r_1} \times ... \times \mathbb{CP}^{r_n} \times \mathbb{C}^m$ for some positive integer m, and so $\pi_*(\pi^*(a_{i_1,...,i_n}))$ is represented by $(\mathbb{CP}^{r_1} \times ... \times \mathbb{CP}^{r_n}) \times \alpha$ with the stable complex structures on each factor, which is the same as $(\mathbb{CP}^{r_1} \times ... \times \mathbb{CP}^{r_n}) a_{i_1...i_n}$ under multiplicating in the complex cobordism ring. Since $\mathbb{CP}^i$ and $a_{i_1...i_n}$ (being a sum of products of the coefficients of the formal group law over $\Omega^*(pt)$) are in the image of h, the element M also lies in the image of h, hence h is surjective, and therefore an isomorphism. ☐

4 Virtual fundamental class correction term

In [3], it is proved that there is a mysterious correction term to the virtual fundamental class that they prove gives rise to an associative generalised quantum product (Lemma 6.5 in [1]). In this section, we work out some results that describe this correction term explicitly. this section, let $\mathbb{E}_*$ be the coefficient ring associated to a counting theory (e.g. complex cobordism or K-theory; for precise definitions, refer to [1]), let $R = \mathbb{E}_*[[D_0, \ldots, D_N]]/\langle D_j^2 - D_j \cdot \tilde{F}(D_j) \rangle$, and $F(x, y)$ is the formal group law associated to it. We also denote by Ix the additive inverse of the element x under the formal group law, so in particular $IF(D_0, D_1, ..., D_{j-1}, \tilde{D}_j, D_{j+1}, ..., D_N)$ is the additive inverse under the formal group law of the sum (again under the formal group law) of all the variables $D_0, ..., D_N$ except for D_j.

We recall first lemma 6.5 in [1].

Lemma 4.1. *There is a power series $f(D_0, D_1, \ldots,) = 1 + \cdots$ for which $F(D_0, \ldots, D_N)$ is given by*

$$\sum_{j=0}^{N} D_j \cdot f(D_0, \ldots, D_{j-1}, 0, \ldots) \tag{101}$$

in the ring R.

The significance of the lemma is that f is a "power series in the Chern classes associated to boundary divisors corresponding to sphere bubbling at the output marked point" ([1]), and one can form the corrected fundamental class $[\mathbb{T}]^{cor} = [\mathbb{T}] \cdot f$ by multiplying the naive virtual fundamental class with f, so that it gives rise to an associative generalized quantum product. In the original proof of Lemma 4.1, the power series f is obtained by inductively applying the identity $D_j^2 = D_j \cdot IF(D_0, D_1, ..., D_{j-1}, \tilde{D}_j, D_{j+1}, ..., D_N)$ in the ring R from index $j = 0$ to $j = N$.

In general, the formula for f can be quite complicated, but in the case $F(x, y) = x + y + axy$, where a is a constant, the power series has a very simple form given by $f(D_0, \ldots, D_N) = 1 + aF(D_0, \ldots, D_N)$, which follows from the inductive formula

$$F(F(D_0, \ldots, D_N), D_{N+1}) = F(D_0, \ldots, D_N) + D_{N+1} + aD_{N+1}F(D_0, \ldots, D_N). \tag{102}$$

In particular, this applies to Morava K-theory $K(n)_p$ at prime $p = 2$ with $n = 1$ if we work modulo powers of order greater than 1 and the case of complex K-theory with the associated formal group law $F(x, y) = x + y - xy$, in which case $f(D_0, D_1, \ldots, D_N) = \prod_{j=0}^{N}(1 - D_j)$

4.1 A combinatorial formula

In this section, we give a recursive combinatorial formula for the correction term to the virtual fundamental class. Consider the following triangular lattice

$$\begin{aligned}
&h_{0,0} \\
&h_{1,0}, h_{1,1} \\
&h_{2,0}, h_{2,1}, h_{2,2} \\
&\cdots,
\end{aligned} \tag{103}$$

where if $i = j$, the term $h_{j,j}$ represents the set of monomials in the expansion of $F(D_0, \ldots, D_j)$ in which D_j is the variable of the highest index j. If $i > j$, the term $h_{i,j}$ represents the term of monomials that contain the variable D_i with the highest index i, which are contributed from resolving monomials with D_j as the variable with the highest index j in the step described in the proof of lemma 6.5 in [1], i.e. these monomials are obtained when we replace D_j^2 by $D_j \cdot F(\overline{D}_0, \ldots, \tilde{\overline{D}}_j, \ldots, \overline{D}_N)$ in the ring R_N, where recall by $\overline{x}$ we denote the inverse of x under the formal group law. In particular, $h_{i,j} \in D_i\mathbb{E}_*[[D_0, \ldots, D_i]]$. Let

$$H_i = \sum_{j=1}^{i} h_{i,j}, \tag{104}$$

it follows that the power series $f(D_0, \ldots, D_N)$ in (4.1) is given by

$$f(D_0, \ldots, D_N) = D_N \frac{H_N(D_0, \ldots, D_{N-1}, \overline{F(D_0, \ldots, D_{N-1})})}{\overline{F(D_0, \ldots, D_{N-1})}}, \tag{105}$$

where the expression makes senses beause $H_N(D_0, \ldots, D_N)$ is divisible by D_N.

Notice that $h_{j,j} = F(D_0, \ldots, D_{j-1})G(D_0, \ldots, D_{j-1}, D_j)D_j$, where recall $F(x, y) = x + y + G(x, y)$ is the equation of the formal group law. Also notice that $h_{1,0} = 0$, and for $j > 1$ and $j > i$, we have the following recursive relation

$$
\begin{aligned}
h_{j,i} =& D_i \frac{H_i(D_0, \ldots, D_{i-1}, \overline{F(D_0, \ldots, \tilde{D}_i, \ldots, D_j)})}{\overline{F(D_0, \ldots, \tilde{D}_i, \ldots, D_j)}} - \\
& D_i \frac{H_i(D_0, \ldots, D_{i-1}, \overline{F(D_0, \ldots, \tilde{D}_i, \ldots, D_{j-1})})}{\overline{F(D_0, \ldots, \tilde{D}_i, \ldots, D_{j-1})}},
\end{aligned} \tag{106}
$$

hence the correction term can be calculated recursively using (106).

4.2 Simplified calculation in the quotient ring

Although the expression of f is in general quite complicated, it has a more tractable form is we pass to the quotient ring $\tilde{R}$ where

$$
\tilde{R} = \mathbb{E}[[D_0, \ldots, D_{N+1}]]_* / \langle D_j^2 - D_j \cdot IF(D_0, \ldots, D_{j-1}) \rangle \tag{107}
$$

is obtained as a quotient ring from the relations

$$
D_j^2 - D_j \cdot IF(D_0, \ldots, D_{j-1}) \tag{108}
$$

as shown in the following theorem.

Theorem 4.1. *The formal group law $F(D_0, D_1, ..., D_N)$ is given by*

$$
D_0 \cdot f(0, ...) + D_1 \cdot f(D_0, 0, ...) + D_2 \cdot f(D_0, D_1, 0, ...) + ... + D_N \cdot f(D_0, ..., D_{N-1}, 0) \tag{109}
$$

in $\tilde{R}$, where

$$
f(D_0, ..., D_N) = 1 + F(D_0, ..., D_N) G(F(D_0, ..., D_N), IF(D_0, ..., D_N)). \tag{110}
$$

Proof. The proof goes by induction. The base case is obviously true for $N = 0$. For the inductive step, observe

$$
\begin{aligned}
& F(D_0, ..., D_N, D_{N+1}) \\
& = F(F(D_0, ..., D_N), D_{N+1}) \\
& = F(D_0, \ldots D_N) + D_{N+1} \\
& + F(D_0, \ldots D_N) D_{N+1} G(F(D_0, ..., D_N), D_{N+1}) \\
& = \sum_{j=0}^{N} D_j \cdot f(D_0, \ldots, D_{j-1}, 0, \ldots) + \\
& (1 + F(D_0, \ldots D_N) G(F(D_0, ..., D_N), D_{N+1})) D_{N+1} \\
& = \sum_{j=0}^{N} D_j \cdot f(D_0, \ldots, D_{j-1}, 0, \ldots) + \\
& (1 + F(D_0, \ldots D_N) G(F(D_0, ..., D_N), IF(D_0, ..., D_N))) D_{N+1} \\
& = \sum_{j=0}^{N+1} D_j \cdot f(D_0, \ldots, D_{j-1}, 0, \ldots)
\end{aligned} \tag{111}
$$

where the second last step uses the identity

$$
D_j^2 = D_j \cdot IF(D_0, D_1, ..., D_{j-1}, \tilde{D}_j). \tag{112}
$$

This completes the proof. ☐

We now show some example calculations using the simplified formula in the quotient ring.

4.2.1 Morava K-theory

Recall that the formal group law for Morava K-theory $K_p(n)$ is given by

$$F(x,y) = x + y - v\sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}x^{ip^{n-1}}y^{(p-i)p^{n-1}}\, in\ \tilde{\mathcal{K}}(n)_p, \tag{113}$$

and the identity $F(x,y) = x + y + xyG(x,y)$ implies

$$G(x,y) = -v\sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}x^{ip^{n-1}-1}y^{(p-i)p^{n-1}-1} \tag{114}$$

in $\mathcal{K}(n)_p$. Let $\overline{\tilde{K}(n)_p}$ be the quotient ring of $\tilde{\mathcal{K}}(n)_p$ by the relations in (108). We have the following results as a direct consequence of (110).

Corollary 4.1. *For Morava K-theory, we have*

$$f(D_0,\ldots,D_N,0,\ldots) = 1 - v\sum_{i=1}^{p-1}\binom{p}{i}(\sum_{j=1}^{N}D_j^{ip^{n-1}})F(\overline{D}_0,\ldots,\overline{D}_N)^{(p-i)p^{n-1}-1}, \tag{115}$$

in $\overline{\tilde{K}(n)_p}$, where the variable $\overline{D_j}$ denotes the additive inverse of D_j under the formal group law associated to the Morava K-theory $K_2(n)$. In the special case of $p = 2$, we have

$$f(D_0,\ldots,D_N) = 1 - v(\sum_{i=0}^{N}D_i^{2^{n-1}})\left(\sum_{\alpha}\prod_{j=0}^{N}\overline{D_j}^{t_{j,\alpha}}\prod_{0\le l,m\le N}(-v(\overline{D_l D_m})^{2^{n-1}})^{t_{lm,\alpha}}\right), \tag{116}$$

where the variables $\ldots,t_{j,\alpha},\ldots,t_{lm,\alpha},\ldots$ correspond to a partition of $2^{n-1}-1$ indexed by α.

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